

Yuljia and the Puzzle Master

Once upon a time (although perhaps not so long ago as you might think) three elf-children named Yuljia, Ardu, and Gacerie lived in a place called Hoio, a land then, as now, full of rolling hills, green fields, lakes, and many forests where elves could live without interference from humans. Their parents, Enjifer and Obert, were the forest-keepers for all the woods and streams in the western realm of Hoio, at least those that men have not spoiled beyond redemption.

No doubt you know all about elves, and there is no need to tell where elves came from, how they live, and why they are so infrequently seen by humans. So, you already know that all elves have commissions ordained by the Council of Elves, whose chief is Galandrathral, the wisest elf of all. Yuljia's charge was the fields, lakes, and streams: she knew the grasses and flowers that grew in the open spaces. She knew where the streams became ponds and rivers and lakes, and she knew when all were as God intended - and when they were not. Ardu knew the forests: he knew the trees by name and which leaves and seeds belonged to each. He had a special gift for making sick trees straight and tall again and some said that he could talk to the trees, and that they loved him especially. Gacerie loved all the animals of the fields and forests. She could talk to animals in their own languages, and her spirit gladdened the hearts of all she met.

It came about that Enjifer and Obert were summoned to a high convocation of the Middle Elves in the ancient city of Unork, leaving Yuljia, Ardu, and Gacerie in the care of their Grandpa and Grandma from the land of Arcolina. Grandpa and Grandma Elf were always glad to visit Hoio and their grand-elves, but especially because it gave them a chance to explore the ancient city of Sochi, built in the long-ago days when the kings of men revered learning and knowledge and eagerly sought the wisdom of the elves. Few men, and fewer elves, remember or visit Sochi in these days, but for Grandpa Elf, whose love of the science of men was accounted a bit strange by most elves, it was a place of wonder and he was always glad to explore the endless halls and theaters where men who loved science and knowledge still lived and worked. Unfortunately, many of the secrets of Sochi had been lost to both elves and men over the years and these were of special interest to Grandpa Elf.

Grandpa and Grandma Elf had not been in Hoio too many days when the elf-children proposed a visit to Sochi. It was a clear, cold day in Hoio, good for a trip, and the children and their grandparents got an early start. Elves walk very quickly when they are traveling, so it wasn't too long after breakfast when they had trekked the two leagues or so to the ancient city. The city's guardians were

among the few men who could recognize elves. They were always glad to see elves on their infrequent visits (although it must be said that they sometimes had a hard time not staring at their unusual visitors). Sochi had many treasures and rooms to explore, and the elves had many adventures which are told in other stories. On this occasion, Yuljia found herself in an ancient hall containing a crystal tower filled with glowing globes, all of different colors and hues. A tablet was set in stone before the tower, its smooth surface inviting inscription with the stylus lying nearby. Above the tablet Yuljia read the words,

**If you know the world of numbers,
You'll know the number of these worlds**

Yuljia studied the puzzle. The globes shimmered and sparkled as light from old lamps played across them. She walked around the tower and could see no difference in the way the globes were packed in the tower. There were too many to be seen, much less counted, and Yuljia thought and thought. How could one count what could not be counted? The only way, she decided, was to use what could be seen to find out about what could not be seen. And so she did. After many minutes of careful thought, she decided on her answer, picked up the stylus, and wrote on the tablet.

“I wonder how I will know if I was right?” she thought to herself.

She didn't have long to wait. No sooner had she stepped away from the tablet than all the globes disappeared from the tower like fog dissolving in the sunlight. In their place appeared an ancient sage arrayed as a magi of the old days. He seemed to wake from sleep, and then spoke, seeming to look directly at Yuljia.

“I am the Puzzle-Master”, he said. “Who has come to solve my puzzle?” he said.

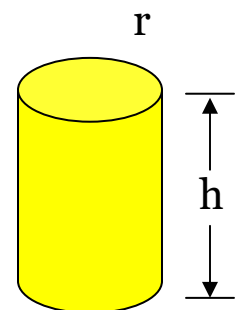
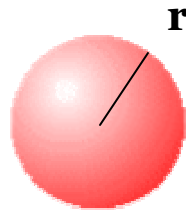
“I have,” said Yuljia, very timidly. “I am Yuljia of the Elves, daughter of Obert and Enjifer, of the line of Galandrathral”.

“You are to be commended, Yuljia of the Elves”, the sage said, “especially for one so young. You have numbered the worlds correctly save two. You have earned the rights of knowledge to my puzzle.”

Then as Yuljia watched, the Puzzle Master's image faded and was replaced by a set of strange and wonderful symbols:

$$\mathbf{Vsphere} := \frac{4}{3} \cdot \pi \cdot \mathbf{r}^3$$

$$\mathbf{Vcyl} := \pi \cdot \mathbf{r}^2 \cdot \mathbf{h}$$



$$N_{\text{spheres}} \leq \frac{\pi}{\sqrt{18}} \cdot \frac{V_{\text{cyl}}}{V_{\text{sphere}}}$$

Yuljia wasn't sure what they meant, but she quickly wrote them down on a piece of parchment to show Grandpa Elf.

When Grandpa Elf saw what Yuljia had discovered he was amazed. He told Grandma Elf that Yuljia had discovered a secret that both men and elves had been seeking for centuries¹ and would lead to much knowledge that could help men to keep their world more beautiful², which of course is something all Elves desire.

"You see", Grandpa Elf said, "if you know the radius of the ball, and the radius and height of the tower, you can calculate the most balls that can fit inside it if they are packed as tightly as possible".

Grandma Elf was amazed that Yuljia had been able to solve the Puzzle-Master's challenge so well, even without knowing the secret formulas, and when Enjifer and Obert returned from Unork, they were just as amazed and decided to take the whole family to the Olive Grove Inn for dinner. They also had important news of their own: Obert had been elected to the Council of the Eldar, those wise and good elves who teach and lead the elf-people in the ways of the One. Yuljia could not remember when she had been more glad to be a daughter of the Elves.

¹ This problem, known as Kepler's Conjecture, has been known since 1631, but only recently solved in 1998, when it was proved that the ratio $\pi/\sqrt{18}$ is the optimum packing density of spheres in a given volume.

² The packing density has application in calculating flow rates of everything from electrons through cylindrical media to water and oil molecules through pipes.